10-6-2000

Chapter 2 Algebraic functions

Functions defined through a combination of $+, -, \cdot, /, \sqrt[n]{\ }$.

2.2 Polynomial functions

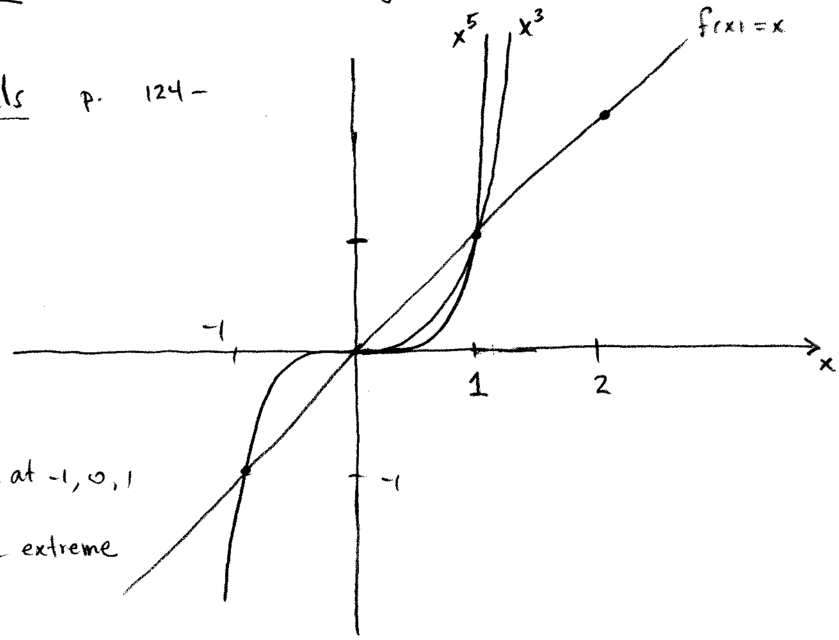
Anything of the form $f(x) = a + bx + cx^2 + dx^3 + \cdots + zx^n$.

Includes constants, lines, quadratics, which we already know.

n is the **degree**, z is the **leading coefficient**.

Know your monomials p. 124 –

Odd degrees

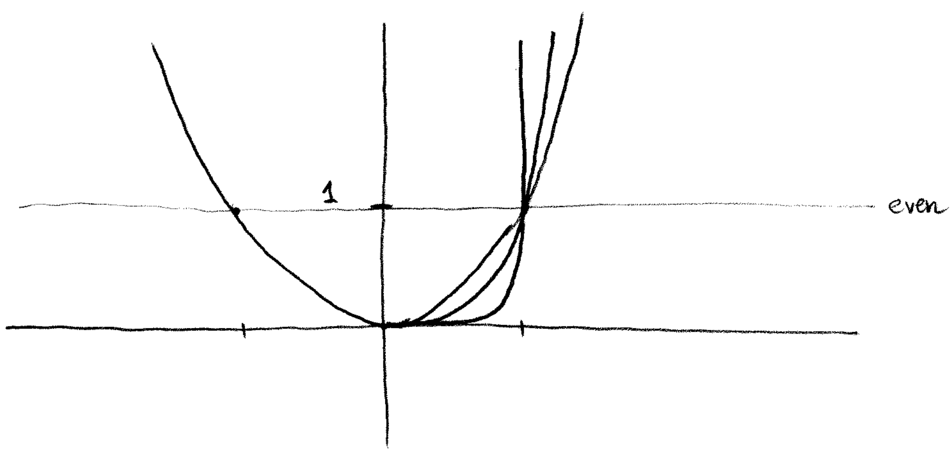


$f(x) = x$

$f(x) = x^3$ same at $-1, 0, 1$

$f(x) = x^5$ more extreme

Even degrees



$f(x) = x^0$

$f(x) = x^2$

$f(x) = x^4$

x^6

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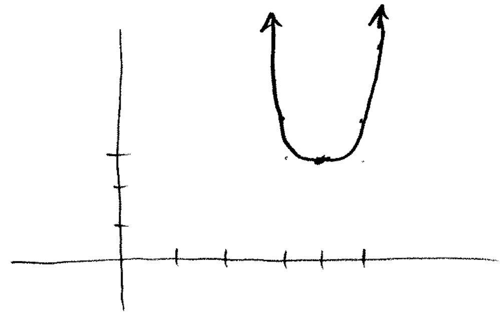
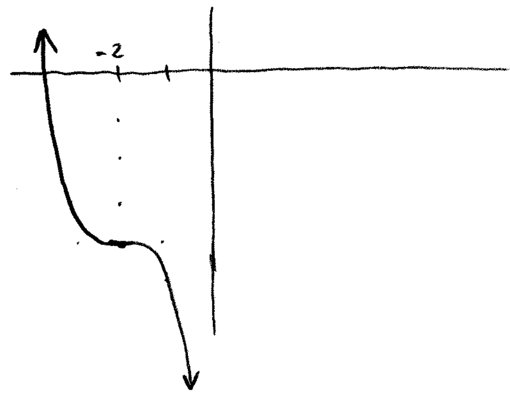
Shifting and scaling monomials

Exercises

$f(x) = (x-4)^6 + 3$

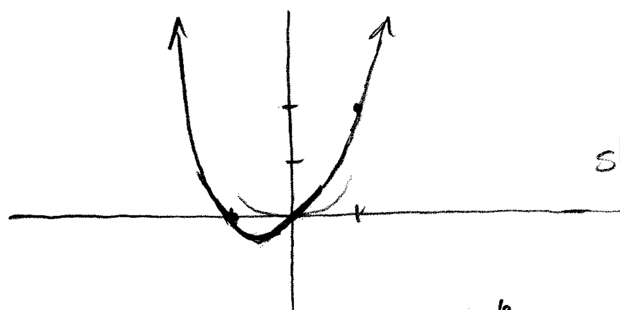


$f(x) = -(x+2)^3 - 4$



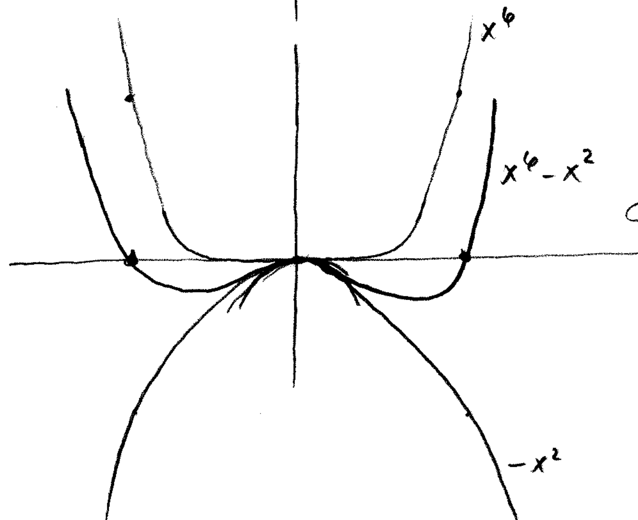
Adding in lower degrees

$f(x) = x^6 + x$



slope/slant at 0

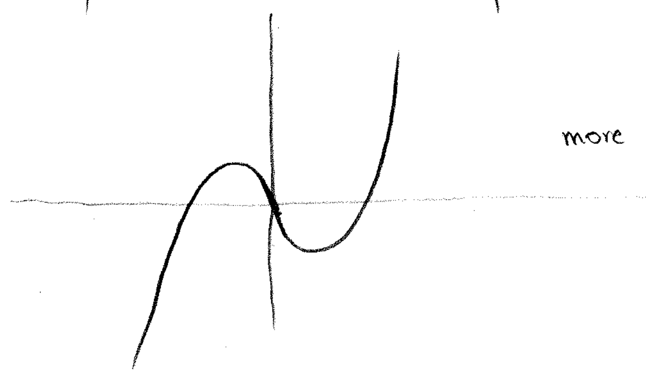
$f(x) = x^6 - x^2$



curvature at 0.

$f(x) = x^3 - 4x$

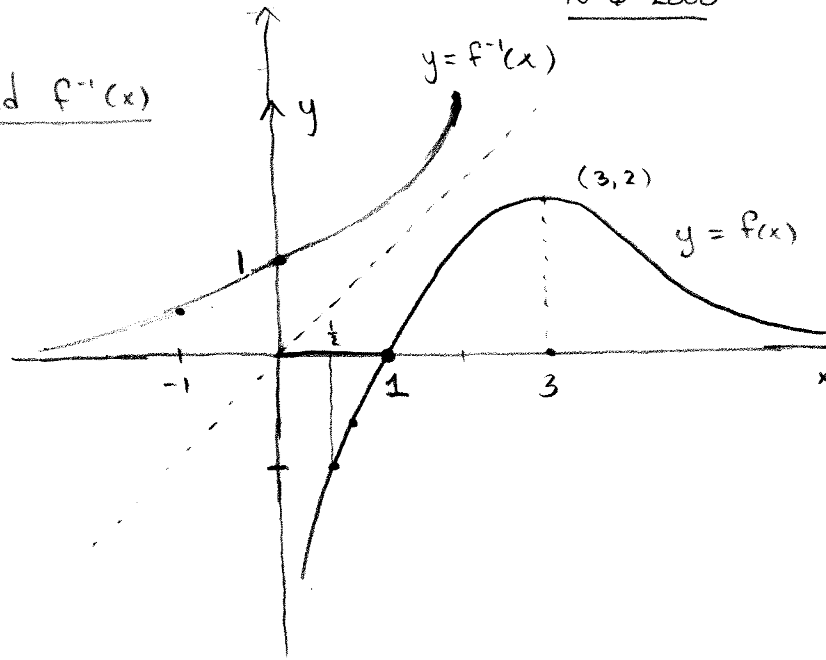
Start at $x = 0$.
$-4x$ dominates.
Move away.
x^3 dominates



more or less.

10-6-2000

Symmetry when graphing $f(x)$ and $f^{-1}(x)$



$(1,0)$ is on the graph of f.

$f(1) = 0$.

Then $f^{-1}(0) = 1$, so $(0,1)$ is on the graph of f^{-1}.

$f(\frac{1}{2}) = -1$,

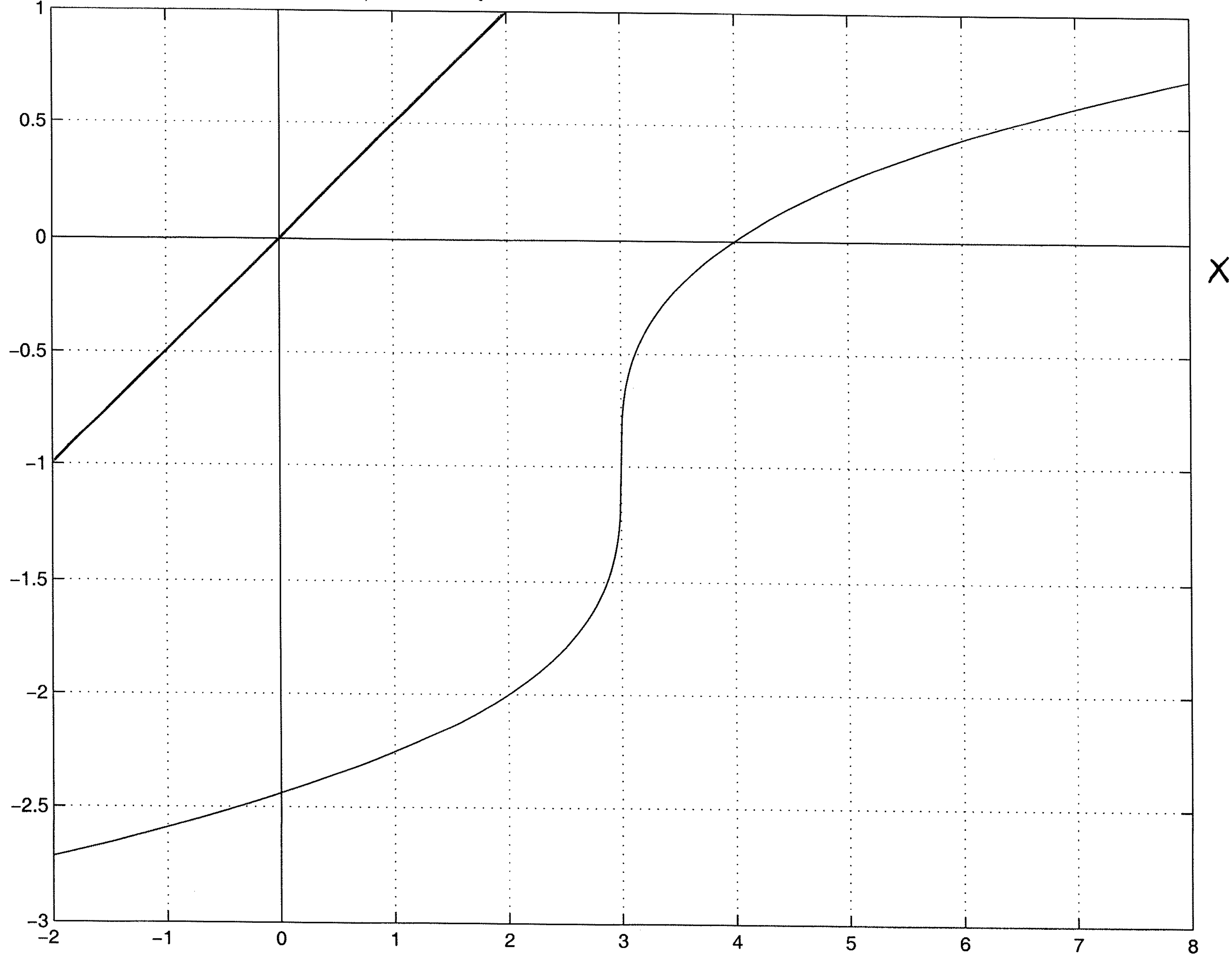
so $f^{-1}(-1) = \frac{1}{2}$.

The function f is not one to one.

One to one: for each y value in the range of f, there is only one corresponding x value.

Function: for each x value, f gives only one y value, called $f(x)$.

$$f(x) = \sqrt[3]{x-3} - 1$$



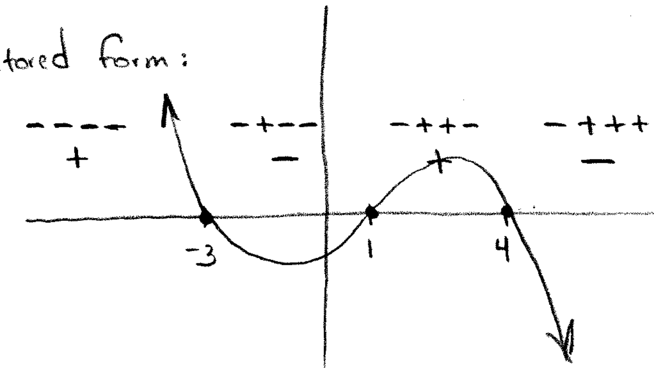
Graphing by using the zeros of a polynomial

We have done this before.

Suppose I have a polynomial in factored form:

Example

$$f(x) = -2(x+3)(x-1)(x-4)$$

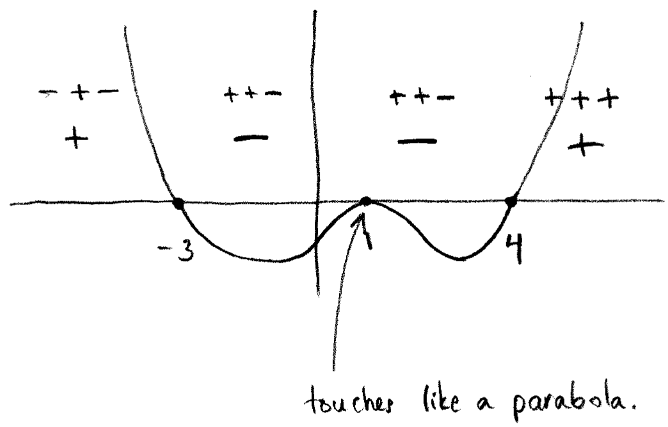


Between the zeros, I need to determine the sign:

End behavior is like $-2x^3$.

Example

$$g(x) = (x+3)(x-1)^2(x-4)$$



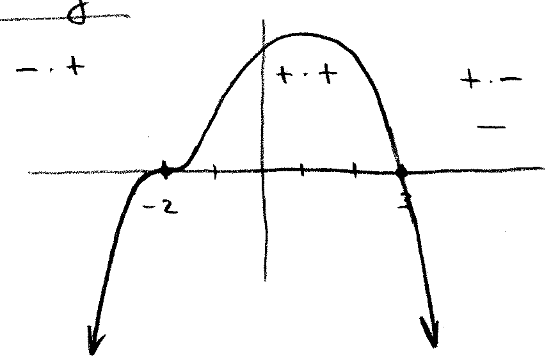
end behavior: like x^4

Near $x = 1$, $f(x) \approx 4(x-1)^2(1-4)$

$$= -12(x-1)^2$$

$x = 1$ is a repeated zero of multiplicity 2

Example $h(x) = (x+2)^3(3-x)$

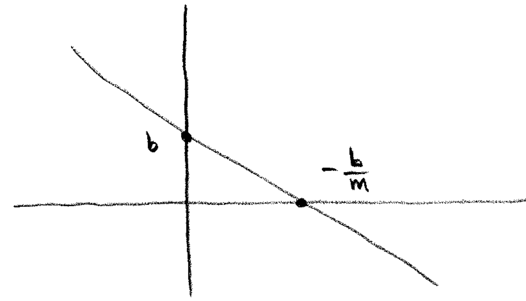


Near $x = -2$, $h(x) \approx (x+2)^3 \cdot 5$

Near $x = 3$, $h(x) \approx 5^3 \cdot (3-x)$

Highest degree 4, negative coefficient.

Finding the zeros of polynomials



Linear:

$$y = mx + b$$

$$0 = mx + b$$

$$x = -\frac{b}{m}$$

Easy!

Quadratic (2)

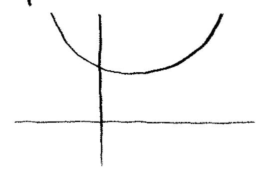
$$y = ax^2 + bx + c$$

$$0 = ax^2 + bx + c$$

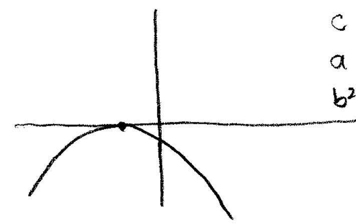
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Harder, still not bad.

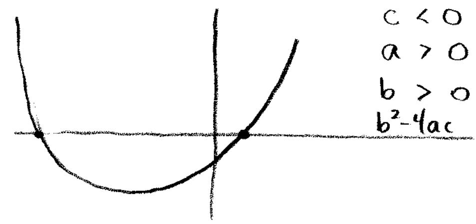
Thought problem:



$c > 0$
$a > 0$
$b^2 - 4ac < 0$
$b < 0$



$c < 0$
$a < 0$
$b^2 - 4ac = 0$



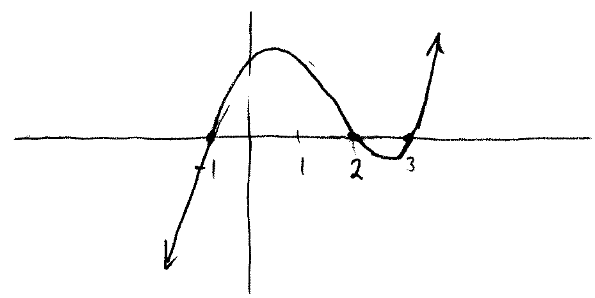
$c < 0$
$a > 0$
$b > 0$
$b^2 - 4ac > 0$

(1) Example $\quad y = x^2 + 3x - 4$

$$= (x+4)(x-1)$$

roots at $x = 1, -4$.

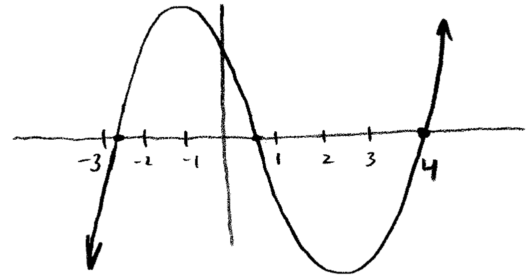
Cubic $\quad y = x^3 - 4x^2 + x + 6 \quad$ Graph:



We should be able to factor this.

In fact, $\quad y = (x-2)(x+1)(x-3)$.

Example $\quad f(x) = x^3 - 2x^2 - 10x + 8$



Graph with your calculator.
One zero appears to be 4,

but the others are not so clear.

Check: $f(4) = 4^3 - 2\cdot 4^2 - 10\cdot 4 + 8 = 64 - 32 - 40 + 8 = 0.$

$(x^3 - 2x^2 - 10x + 8) = (x-4)\cdot(\text{quadratic})$ "Factor theorem"

A divison problem. What quadratic polynomial will work?

$$
\begin{array}{r|llll}
 & x^2 & +2x & -2 & \\
\hline
x-4 & x^3 - 2x^2 & -10x & +8 & \\
 & x^3 - 4x^2 & & & \\
\hline
 & 2x^2 & -10x & +8 & \\
 & 2x^2 & -8x & & \\
\hline
 & & -2x & +8 & \\
 & & -2x & +8 & \\
\hline
 & & & 0 &
\end{array}
$$

match up the pieces in stages.

$f(x) = (x-4)(x^2 + 2x - 2)$

Can we factor further?

Use the quadratic formula:

$x^2 + 2x - 2 = 0$ when $\quad x = \dfrac{-2 \pm \sqrt{4 - 4\cdot 1\cdot(-2)}}{2}$

$= -1 \pm \dfrac{\sqrt{12}}{2}$

$= -1 \pm \dfrac{\sqrt{4}\cdot\sqrt{3}}{2}$

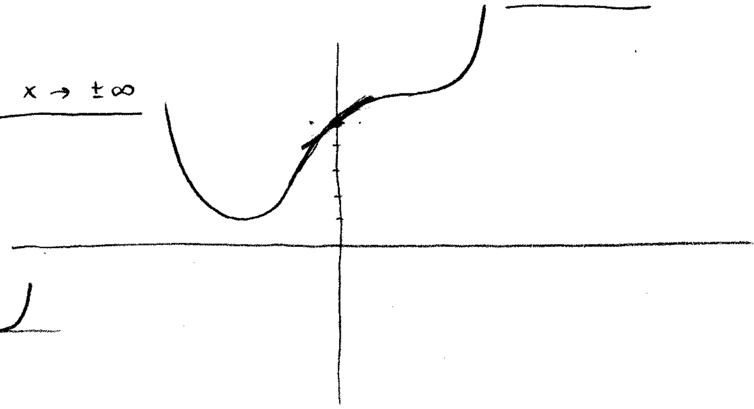
$= -1 \pm \sqrt{3}$

$= -2.73205\ldots \quad , +0.73205\ldots$

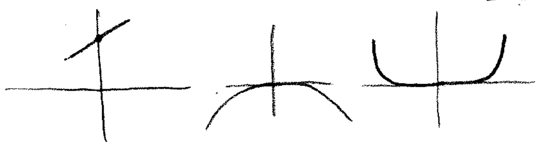
Zeros at $\quad 4, \ -1 + \sqrt{3}, \ -1 - \sqrt{3}.$

$f(x) = (x-4)(x - (-1+\sqrt{3}))(x - (-1-\sqrt{3})).$ completely factored

Behavior of polynomials near $x = 0$ and as $x \to \pm\infty$

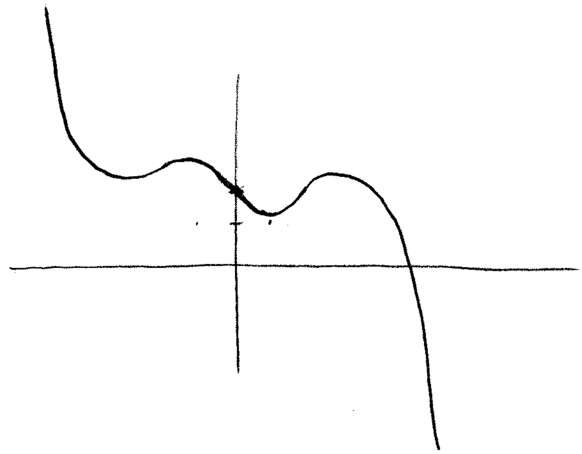


- $f(x) = 5 + x - \frac{1}{2}x^2 + x^4$



Exercises

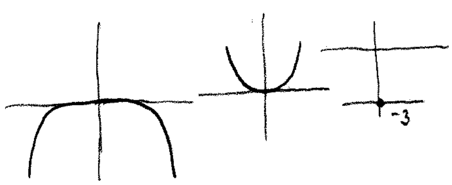
- $f(x) = -x^7 + 5x^3 - x + 2$

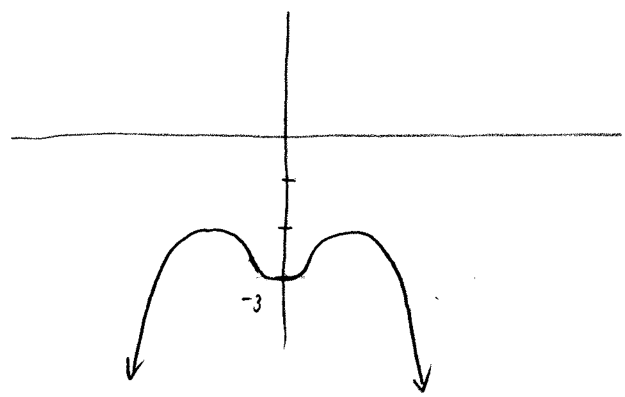


Is f odd? No.

But $f - 2$ is.

- $f(x) = -x^6 + x^2 - 3$





Exercise Let $f(x) = x^3 + x^2 - 26x - 60$.

Find the zeros exactly.

Calculator: one zero at $x = -3$.

$$x^3 + x^2 - 26x - 60 = (x+3) \cdot \text{quadratic}$$

$$\begin{array}{r|l}
 & x^2 - 2x - 20 \\
\hline
x+3 & x^3 + x^2 - 26x - 60 \\
 & \underline{x^3 + 3x^2} \\
 & -2x^2 - 26x - 60 \\
 & \underline{-2x^2 - 6x} \\
 & -20x - 60 \\
 & \underline{-20x - 60} \\
 & 0
\end{array}$$

$$x^3 + x^2 - 26x - 60 = (x+3)(x^2 - 2x - 20)$$

Other roots:

$$x = \frac{2 \pm \sqrt{4 - 4(1)(-20)}}{2}$$

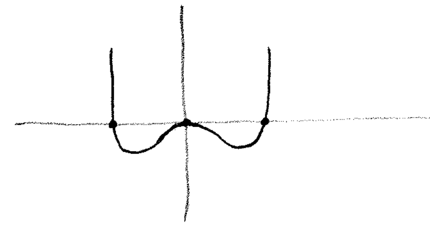
$$= 1 \pm \tfrac{1}{2}\sqrt{84}$$

$$= 1 \pm \tfrac{1}{2}\sqrt{4}\sqrt{21}$$

$$= 1 \pm \sqrt{21}$$

Roots $-3,\ 1+\sqrt{21},\ 1-\sqrt{21}$

$\approx -3,\ 5.58257\ldots,\ -3.58257\ldots$



Example Factor $x^4 - 3x^2$

$$= x^2(x^2-3)$$

$$= x^2(x-\sqrt{3})(x+\sqrt{3})$$

Zeros $x = 0, 0, \sqrt{3}, -\sqrt{3}$

Example Factor $x^4 - 3x^2 + 2$

$$= (x^2 - 1)(x^2 - 2)$$

$$= (x+1)(x-1)(x+\sqrt{2})(x-\sqrt{2})$$

Example Factor $x^4 - 3x^2 + 1$

Harder.

Think of $z = x^2$.

We need to factor (or solve) $z^2 - 3z + 1 = 0$

$$z = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

Thus, $x = \pm\sqrt{z} = \pm\sqrt{\frac{3 \pm \sqrt{5}}{2}}$.

The solutions are: $x = \sqrt{\frac{3+\sqrt{5}}{2}}, \sqrt{\frac{3-\sqrt{5}}{2}}, -\sqrt{\frac{3-\sqrt{5}}{2}}, -\sqrt{\frac{3+\sqrt{5}}{2}}$.

They're exact!

Rational zero test

Polynomial with integer coefficients: $2x^4 + 3x^3 - x^2 + 6$.

Some zeros may be rational (including integers), but we can narrow down the possibilities.

2 has factors 1, 2 — possible denominators

6 has factors 1, 2, 3, 6 — possible numerators

Possible rational roots

The end of the road with factoring

- not all quadratics can be factored into real linear factors

Example $x^2+1=0$

$$x = \frac{0 \pm \sqrt{0^2-4}}{2} = \pm\sqrt{-1} = \pm i$$

$$x^2+1 = (x-\sqrt{-1})(x+\sqrt{-1}) = (x-i)(x+i)$$

Complex numbers are awesome, but not done in this class.

However, every polynomial can be factored into real linear and/or quadratic factors — the Fundamental Theorem of Algebra, proved by Gauss in 1799. (see p. 178).

- not all roots are integers, rationals, or things with square roots

Example $x = \pm\sqrt{\frac{3\pm\sqrt{5}}{2}}$ in the earlier example

Example $x^3-5=0$

$$x = \sqrt[3]{5}$$